RJEŠENJE PISMENOG ISPITA, MATEMATIKA 1, 28.08.2017.

A grupa:

1. Zadani su vektori $\vec{a} = (1,1,4)$ i $\vec{b} = (-2,3,0)$ i $\vec{c} = (-\lambda, -3\lambda, 1)$.

   (a) Nađite parametar λ tako da $\vec{c}$ bude okomit na vektor $2\vec{a} - \vec{b}$. (1)

   $\vec{c} \perp (2\vec{a} - \vec{b}) \Leftrightarrow \vec{c} \cdot (2\vec{a} - \vec{b}) = 0$

   $(-\lambda, -3\lambda, 1) \cdot (4, -1, 8) = -4\lambda + 3\lambda + 8 = 0 \Rightarrow \lambda = 8$

   (b) Izračunajte površinu trokuta razapetog vektorima $\vec{a}$ i $\vec{b}$. (1)

   $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 4 \\ -2 & 3 & 0 \end{vmatrix} = (-12, -8, 5) \Rightarrow |\vec{a} \times \vec{b}| = \sqrt{144 + 64 + 25} = \sqrt{233}$$

   $$P = \frac{\sqrt{233}}{2}$$

2. Zadana je funkcija $f(x) = \sqrt[3]{2x} \cdot \ln(-x-1) + \sqrt{-x^2 - x + 12}$.

   (a) Odredite domenu funkcije f. (1)

   Uvjeti:

   - $-x - 1 > 0 \Rightarrow x < -1$
   - $-x^2 - x + 12 \geq 0$ (skicirajte parabolu) $\Rightarrow x \in [-4, 3]$

   Domena funkcije je $\mathcal{D}_f = [-4, -1\rangle$.

   (b) Izračunajte $f'(-2)$. (1)

   $$\begin{aligned} f'(x) &= (\sqrt[3]{2x})' \cdot \ln(-x-1) + \sqrt[3]{2x} \cdot (\ln(-x-1))' + (\sqrt{-x^2-x+12})' \\ &= \frac{1}{3}(2x)^{-\frac{2}{3}}\, 2\ln(-x-1) + \sqrt[3]{2x}\frac{-1}{-x-1} + \frac{-2x-1}{2\sqrt{-x^2-x+12}} \\ &= \frac{2}{3\sqrt[3]{4x^2}}\ln(-x-1) + \frac{\sqrt[3]{2x}}{x+1} - \frac{2x+1}{2\sqrt{-x^2-x+12}} \end{aligned}$$

   $$f'(-2) = -\sqrt[3]{-4} + \frac{3}{2\sqrt{10}}$$

3. Odredite jednadžbe tangenti na graf funkcije $f(x) = \operatorname{arc\,tg}(2x-1)$ koje su okomite na pravac $y + 5x - 3 = 0$. (2)

   Koeficijent smjera pravca je -5, dakle koeficijent smjera tangenti mora biti $\frac{1}{5}$. Budući da je koeficijent smjera tangenti derivacija u točki dirališta mora vrijediti:

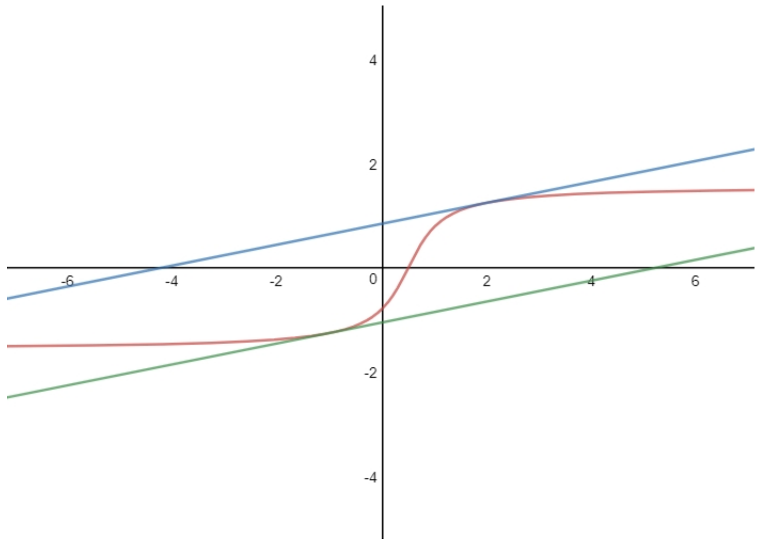
   $$f'(x) = \frac{2}{1 + (2x-1)^2} = \frac{1}{5} \Rightarrow (2x-1)^2 = 9$$

   $$2x - 1 = \pm 3, \;\; x_1 = 2, x_2 = -1$$

Dakle točke dirališta su $T_1(2, \operatorname{arc\,tg}3)$ i $T_2(-1, \operatorname{arc\,tg}(-3))$. Jednadžbe tangenti su:

$$t_1 \ldots\ y = \frac{1}{5}x - \frac{2}{5} + \operatorname{arc\,tg}3$$

$$t_2 \ldots\ y = \frac{1}{5}x + \frac{1}{5} + \operatorname{arc\,tg}(-3)$$



4. Riješite integrale: (2)

(a)

$$\int \frac{e^{3x}}{(e^x-1)^2}\,\mathrm{d}x = \begin{bmatrix} t = e^x - 1 \\ dt = e^x\,\mathrm{d}x \\ e^x = t+1 \end{bmatrix} = \int \frac{(t+1)^3}{t^2}\frac{\mathrm{d}t}{t+1} = \int \frac{(t+1)^2}{t^2}\,\mathrm{d}t$$

$$= \int \frac{t^2+2t+1}{t^2}\,\mathrm{d}t = \int \left(1 + \frac{2}{t} + t^{-2}\right)\,\mathrm{d}t$$

$$= t + 2\ln|t| - \frac{1}{t} + c$$

$$= e^x - 1 + 2\ln|e^x - 1| - \frac{1}{e^x-1} + c, c \in \mathbb{R}$$

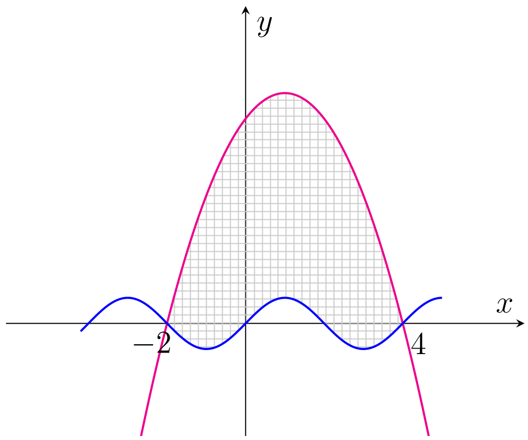
(b) $\displaystyle\int \frac{7x+2}{(x-1)(x+2)}\,\mathrm{d}x$

Podintegralna funkcija je racionalna pa ju prvo treba rastaviti na parcijalne razlomke:

$$\frac{7x+2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \Rightarrow A = 3, B = 4.$$

$$\int \frac{7x+2}{(x-1)(x+2)}\,\mathrm{d}x = \int \left(\frac{3}{x-1} + \frac{4}{x+2}\right)\,\mathrm{d}x = 3\ln|x-1| + 4\ln|x+2| + c, c \in \mathbb{R}$$

5. Skicirajte i izračunajte površinu lika definiranog s $y \geq \sin(\frac{\pi}{2}x)$ i $y \leq -x^2 + 2x + 8$. (2)



$$P = \int_{-2}^{4} \left(-x^2 + 2x + 8 - \sin(\frac{\pi}{2}x) \right)\, \mathrm{d}x = \left[-\frac{x^3}{3} + x^2 + 8x + \frac{2}{\pi}\cos(\frac{\pi}{2}x) \right] \Bigg|_{-2}^{4} = 36 + \frac{4}{\pi}$$

B grupa:

1. Zadani su vektori $\vec{a} = (2, 1, -1)$ i $\vec{b} = (3, 3, 0)$ i $\vec{c} = (\lambda, -\lambda, 5)$.

   (a) Nađite parametar λ tako da $\vec{c}$ bude okomit na vektor $\vec{a} - 2\vec{b}$. (1)

   $\vec{c} \perp (\vec{a} - 2\vec{b}) \Leftrightarrow \vec{c} \cdot (\vec{a} - 2\vec{b}) = 0$

   $(\lambda, -\lambda, 5) \cdot (-4, -5, -1) = -4\lambda + 5\lambda - 5 = 0 \Rightarrow \lambda = 5$

   (b) Izračunajte površinu trokuta razapetog vektorima $\vec{a}$ i $\vec{b}$. (1)

   $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 3 & 3 & 0 \end{vmatrix} = (3, -3, 3) \Rightarrow |\vec{a} \times \vec{b}| = \sqrt{9 + 9 + 9} = 3\sqrt{3}$$

   $$P = \frac{3\sqrt{3}}{2}$$

2. Zadana je funkcija $f(x) = \ln(5 - x) \cdot \sqrt{-x^2 + 7x - 6} - \cos(\pi x)$.

   (a) Odredite domenu funkcije f. (1)

   Uvjeti:

   - $5 - x > 0 \Rightarrow x < 5$
   - $-x^2 + 7x - 6 \geq 0$ (skicirajte parabolu) $\Rightarrow x \in [1, 6]$

   Domena funkcije je $\mathcal{D}_f = [1, 5\rangle$.

   (b) Izračunajte $f'(2)$. (1)

   $$\begin{aligned} f'(x) &= (\ln(5 - x))' \cdot \sqrt{-x^2 + 7x - 6} + \ln(5 - x) \cdot (\sqrt{-x^2 + 7x - 6})' - (\cos(\pi x))' \\ &= \frac{-1}{5 - x}\sqrt{-x^2 + 7x - 6} + \ln(5 - x)\frac{-2x + 7}{2\sqrt{-x^2 + 7x - 6}} + \sin(\pi x) \cdot \pi \end{aligned}$$

   $$f'(2) = -\frac{2}{3} + \frac{3}{4}\ln 3$$

3. Odredite jednadžbe tangenti na graf funkcije $f(x) = \operatorname{arc\,ctg}(3x - 2)$ koje su okomite na pravac $3y - 2x - 1 = 0$. (2)

   Koeficijent smjera pravca je $\frac{2}{3}$, dakle koeficijent smjera tangenti mora biti $-\frac{3}{2}$. Budući da je koeficijent smjera tangenti derivacija u točki dirališta mora vrijediti:

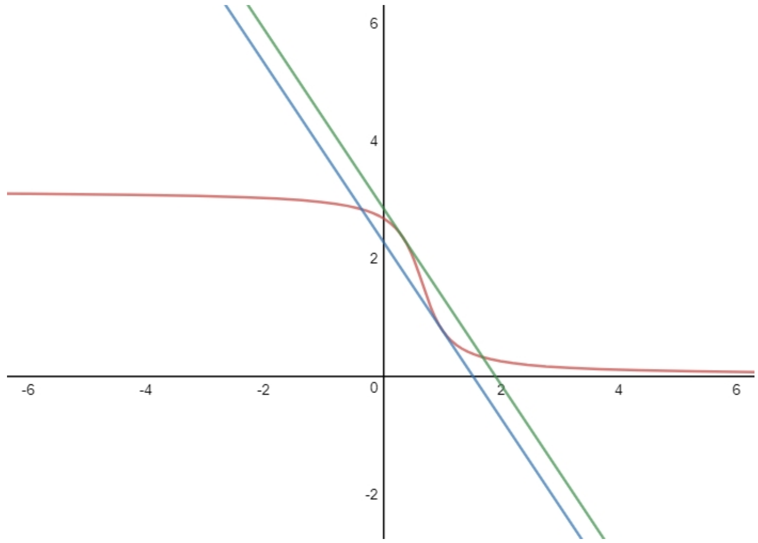
   $$f'(x) = \frac{-3}{1 + (3x - 2)^2} = \frac{-3}{2} \Rightarrow (3x - 2)^2 = 1$$

   $$3x - 2 = \pm 1, \;\; x_1 = 1, x_2 = \frac{1}{3}$$

   Dakle točke dirališta su $T_1(1, \operatorname{arc\,ctg} 1)$ i $T_2(\frac{1}{3}, \operatorname{arc\,ctg}(-1))$. Jednadžbe tangenti su:

   $$t_1 \ldots \; y = -\frac{3}{2}x + \frac{3}{2} + \operatorname{arc\,ctg} 1$$

   $$t_2 \ldots \; y = -\frac{3}{2}x + \frac{1}{2} + \operatorname{arc\,ctg}(-1)$$



4. Riješite integrale: (2)

(a)

$$\int 2^x(1-2^{2x})^2\,\mathrm{d}x = \begin{bmatrix} t = 2^x \\ dt = 2^x \ln 2\,\mathrm{d}x \end{bmatrix} = \int (1-t^2)^2 \frac{\mathrm{d}t}{\ln 2}$$
$$= \frac{1}{\ln 2}\int (1-2t^2+t^4)\,\mathrm{d}t$$
$$= \frac{1}{\ln 2}\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right) + c$$
$$= \frac{1}{\ln 2}\left(2^x - \frac{2}{3}2^{3x} + \frac{1}{5}2^{5x}\right) + c, c \in \mathbb{R}$$

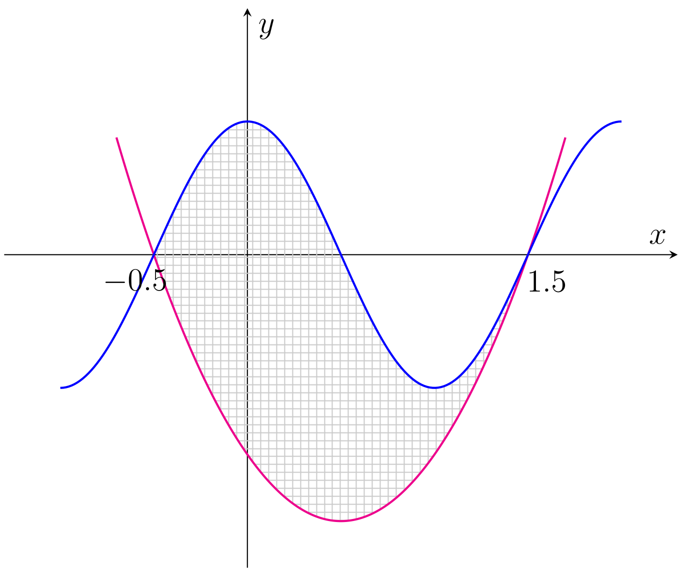
(b) $\displaystyle\int \frac{x-2}{(x+1)(x+4)}\,\mathrm{d}x$

Podintegralna funkcija je racionalna pa ju prvo treba rastaviti na parcijalne razlomke:

$$\frac{x-2}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4} \Rightarrow A = -1, B = 2.$$

$$\int \frac{x-2}{(x+1)(x+4)}\,\mathrm{d}x = \int \left(\frac{-1}{x+1} + \frac{2}{x+4}\right)\,\mathrm{d}x = -\ln|x+1| + 2\ln|x+4| + c, c \in \mathbb{R}$$

5. Skicirajte i izračunajte površinu lika definiranog s $y \leq \cos(\pi x)$ i $y \geq 2x^2 - 2x - \frac{3}{2}$. (2)



$$P = \int_{-\frac{1}{2}}^{\frac{3}{2}} \left(\cos(\pi x) - 2x^2 + 2x + \frac{3}{2} \right) \,\mathrm{d}x = \left[\frac{1}{\pi} \sin(\pi x) - \frac{2}{3}x^3 + x^2 + \frac{3}{2}x \right] \Bigg|_{-\frac{1}{2}}^{\frac{3}{2}} = \frac{8}{3}$$