OPTIMAL CONSTRAINED COVARIANCE CONTROL (OC^3) POSTOPTIMAL ANALYSIS

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ABSTRACT: *Using the optimal constrained covariance control technique (OC^3) the disadvantages of the classical LQG optimal control technique are avoided. Namely, the classical Linear Quadratic Gaussian (LQG) optimal control approach is impractical for two reasons. First, there are usually no physical sense of the performance weighting matrices Q and R, second, the objectives are conflicting and there is no design that is best with respect to all objectives. Thus, a very difficult iterative design procedure must be applied to determine the necessary optimal control performance criterion. Using the proposed technique these disadvantages are avoided. A procedure for the postoptimal analysis of the constrained covariance control is proposed. In the paper, one illustrative example is presented.*

Key Words: *LQG control, constrained covariance control, postoptimal analysis*

1 INTRODUCTION

The main motivation for Optimal Constrained Covariance Control (OC^3) is that many real control systems have performance requirements naturally stated in terms of the root-mean-square (RMS) values. These requirements are usually given in the form of inequality constraints. The optimal control problem is characterized by compromises and tradeoffs, with performance requirements and magnitude of the input energy [1]. For example, the objective of a dynamic positioning system is to maintain the position and heading of a vessel at reference values with acceptable accuracy [2]. The design of the systems involves a compromise between the accuracy of holding a position and the need to suppress excessive thruster response. The classical Linear Quadratic Gaussian (LQG) optimal control approach is impractical for two reasons. First, there are usually no physical sense of weighting matrices Q and R, second, a very difficult iterative design procedure must be applied to determine the necessary optimal control performances [3]. The use of Covariance Control [4] by procedure of assigning the state covariance has theoretical meaning, only. Namely, the assigning of complete covariance matrix is very hard requirement in a lot of real engineering systems. Apart, the procedure for assigning the desired cross-correlation terms is usually unwieldy (particularly for large-order systems). The requirements in the form of inequality constraints (some of diagonal terms) are more acceptable. The optimal linear controller is designed with OC^3 technique in such a way that the specified state covariance of a closed loop system is below the ordered ones. It is achieved with minimum input energy. Next, the suggested procedure holds the original convexity of LQ problem. The application of proposed method for dynamic positioning control system is given in [5].

2 OPTIMAL CONSTRAINED COVARIANCE CONTROL (OC^3)

Consider the continuos linear system described by:

$$\begin{aligned} \dot{\boldsymbol{x}}(\mathrm{t}) &= \boldsymbol{A}\boldsymbol{x}(\mathrm{t}) + \boldsymbol{B}\boldsymbol{u}(\mathrm{t}) + \boldsymbol{G}\boldsymbol{w}(\mathrm{t}) \\ \boldsymbol{y}(\mathrm{t}) &= \boldsymbol{C}\boldsymbol{x}(\mathrm{t}) + \boldsymbol{v}(\mathrm{t}) \end{aligned} \tag{1}$$

where **x** is the n-dimensional state vector, **u** is the p-dimensional input vector, **y** is the r-dimensional output vector, and **w** and **v** are Gaussian white noise with zero mean and covariance matrices $\mathbf{R}_w$ and $\mathbf{R}_v$, respectively. The required performances are given in the form of inequality constraints:

$$\mathrm{diag}(\boldsymbol{D}_x) \leq \boldsymbol{d}_0 \tag{2}$$

where $\mathbf{D_x}$ is the state covariance matrix of closed loop system and $\mathbf{d_0}$ is desired upper limit for diagonal elements of $\mathbf{D_x}$. The cost function (price) is given in the form:

$$\mathrm{J} = \mathrm{trace}(\boldsymbol{R}\boldsymbol{D}_u) \tag{3}$$

where $\mathbf{D_u}$ is the control input covariance matrix of closed loop system and **R** is weighting matrix. $\mathbf{D_x}$ and $\mathbf{D_u}$ are defined as:

$$\begin{aligned} \boldsymbol{D}_x &= \mathrm{E}\left[\boldsymbol{x}(\mathrm{t})\boldsymbol{x}(\mathrm{t})^{\mathrm{T}}\right] \\ \boldsymbol{D}_u &= \mathrm{E}\left[\boldsymbol{u}(\mathrm{t})\boldsymbol{u}(\mathrm{t})^{\mathrm{T}}\right] \end{aligned} \tag{4}$$

First, we define weighting matrix **Q** as:

$$\boldsymbol{Q} = \boldsymbol{X}\boldsymbol{X}^{\mathrm{T}} \tag{5}$$

where **X** is arbitrary matrix and **Q** is always symmetric and positive semi-definite matrix. The terms of **X** are variables in optimisation problem. In that way, a local minimizer is defined as the well-known LQR problem:

$$\boldsymbol{K} = \mathrm{lqr}(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Q}, \boldsymbol{R}) \tag{6}$$

The stationary covariance matrices of the estimate and the input can be computed from:

$$
\begin{aligned}
&(\boldsymbol{A}-\boldsymbol{BK})\boldsymbol{D_x} + \boldsymbol{D_x}(\boldsymbol{A}-\boldsymbol{BK})^{\boldsymbol{T}} + \boldsymbol{GQ_w G^T} = 0 \\
&\boldsymbol{D_u} = \boldsymbol{KD_x K^T}
\end{aligned}
\tag{7}
$$

In this way the algorithm is based on solving a sequence of standard linear quadratic control problems [6]. The procedure uses the Sequential Quadratic Programming (SQP) method. Without proof it is clear that the suggested procedure holds the original convexity of the LQR problem. There is no problem to supply the SQP algorithm with the analytically defined gradients of the cost function (3) and constraints (2).

3 POSTOPTIMAL ANALYSIS

When the optimisation procedure is finished, the sensitivity of the solutions to desired system performances, model inaccuracies and other initial conditions have to be analysed. This analysis is known as postoptimal analysis [7]. When the sensitivity of solutions to desired system performances is of our concern, then it can be shown that under particular circumstances, a slight change of desired system performances could significantly improve the optimal solution value. Namely, a slight relaxation of desired system position accuracy could result with significant energy savings. As part of postoptimal analysis the possibilities of price-performance improvements can be tested. The optimisation problem can be given in the general form by:

$$
\begin{aligned}
&\min_x f^0(x) \\
&f^i(x) \le \theta_i \qquad i = 1,\ldots,p
\end{aligned}
\tag{8}
$$

where:

$f^0(x)$ - cost (price) function,
$f^i(x)$ - constraint function,
θ_i - constraint (performance) value.

The above optimisation is easy to explain. The cost function represents the price of realisation (such as energy consumption), while the constraint function represents the desired technical performances of our system (such as desired position accuracy). The corresponding augmented Lagrange function is:

$$
L(x,\lambda,\theta) = f^0(x) + \sum_{i=1}^{p} \lambda_i (f^i(x) - \theta_i) \tag{9}
$$

Assuming that Slater's condition [7] is valid for some point x^* and Θ^*, then:

$$
\frac{\partial f^0(x^*,\theta^*)}{\partial \theta_i^*} = -\lambda_i^* \tag{10}
$$

Equation (10) can be interpreted as the shadow price. This term is often used in economics when the optimal solutions are sought. It gives the relation for the sensitivity of solution to small change of constrained value (8). A small (or zero) value of Lagrange multiplier indicates that a slight change in this constraint does not have influence on the cost function. On the other hand a large value of Lagrange multiplier indicates that the corresponding optimal value of the cost function is more susceptible to changes in this constraint. In the case of OC^3 design, equation (10) expresses the cost sensitivity related to the slight change of control system accuracy performances. However, sometimes the normed equation (10) is preferred, and is given by:

$$s_i = \frac{\dfrac{\partial f^{0}(x^*,\theta^*)}{f^{0}(x^*,\theta^*)}}{\dfrac{\partial \theta_i^*}{\theta_i^*}} = -\lambda_i^* \frac{\theta_i^*}{f^{0}(x^*,\theta^*)} \tag{11}$$

Here the relative change of the cost function optimal value and constrained values are used. Parameter s_i represents the normed shadow price.

4 ILUSTRATIVE EXAMPLE - RUDDER ROLL CONTROL

The consequences of roll motions during ship operations can seriously degrade the performance of working effectiveness. The rudder is primarily used to create torques to turn the ship into a new course, but at the same time roll torque is generated, too. This second effect from the rudder can be utillised to obtain the damping of roll motion simultaneously with the control of the ship course. The rudder roll stabilisation (RRS) approach is attractive since the existing equipment can be used, and thus it is a relatively inexpensive solution [8].

4.1 Ship Model

The linear ship models (yaw, roll) are described in the form of transfer functions [8]:

$$\begin{aligned} \phi(s) &= h_\phi(s)\left[K_{dp}\delta(s) + K_{vp}v(s) + w_\phi(s)\right] \\ \psi(s) &= h_\psi(s)\left[K_{dr}\delta(s) + K_{vr}v(s) + w_\psi(s)\right] \end{aligned} \tag{12}$$

where ϕ, ψ are roll and yaw angle, respectively, δ is the rudder angle, w_ϕ, w_ψ are coloured noise describing the wave motion, and auxiliary variables are defined as:

$$h_\varphi(s) = \frac{\omega_n^2}{s^2 + 2\varsigma_n\omega_n s + \omega_n^2}$$

$$h_\psi(s) = \frac{1}{(1+\tau_r s)s}$$

$$v(s) = \frac{K_{dv}}{(1+\tau_{vr}s)}\delta(s)$$

The new state variable v represent the sway velocity induced by the rudder motion alone. A more detailed explanation of different parameters is given in [8].

4.2 Wave model

The ship motion is determined by the rudder and environmental disturbances. For the rudder roll stabilisation, only high-frequency roll motion can be reduced. These disturbances can be simulated using a second order linear approximation of the Pierson-Moskowitz spectral density function [9]. The coloured noise transfer functions are in the form:

$$h_i(s) = \frac{2\varsigma_0\sigma_i}{s^2 + 2\varsigma_0\omega_0 s + \omega_o^2} \qquad i = 1,2 \tag{13}$$

where σ_i, i=1,2, are constants describing the wave intensity, ζ_0 is relative damping coefficient and ω_0 is the dominating wave frequency. Then, the disturbances, w_ϕ and w_ψ are given by:

$$w_\psi(s) = h_1(s)w_1(s), \qquad w_\varphi(s) = h_2(s)w_2(s) \tag{14}$$

where w_1 and w_2 are Gaussian white noise.

4.3 Rudder model

The steering machine is highly nonlinear, and in RRS modelling the dominant nonlinearities are magnitude and rate saturation [10]. The model of the steering machine is shown in Fig. 1.

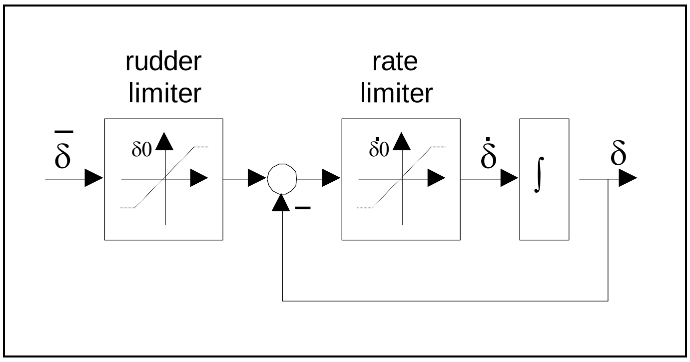

Fig.1. *Model of stearing machine*

The mathematical model of the stearing machine is in the form:

$$\dot{\delta} = \frac{1}{T_\delta} sat(sat(\bar{\delta}, \delta_0) - \delta, \dot{\delta}_0) \tag{15}$$

where δ_0 and $\dot{\delta}_0$ are magnitude and rate saturation parameters, and T_δ is time constant.

4.4 Postoptimal analysis

The proposed method of postoptimal analysis is applied to the rudder roll stabilisation of ships, given in [8]. Only the linear motion is analysed. The parameters for the ship model (12) are taken from [10]. The cost function in the postoptimal analysis was chosen to be the rudder activity, while the constraint function was the RMS value of roll deviation. The results of the postoptimal analysis are given in Fig. 2. It can be seen that the (shadow price) parameter s_i is approximately 1 until the RMS value of roll deviation becomes 1 [deg^2]. After that the shadow price value steeply rises. The interpretation of this example from the economic aspect is that there is the price to be paid if we insist to have the RMS value of roll deviation better than 1 [deg^2].

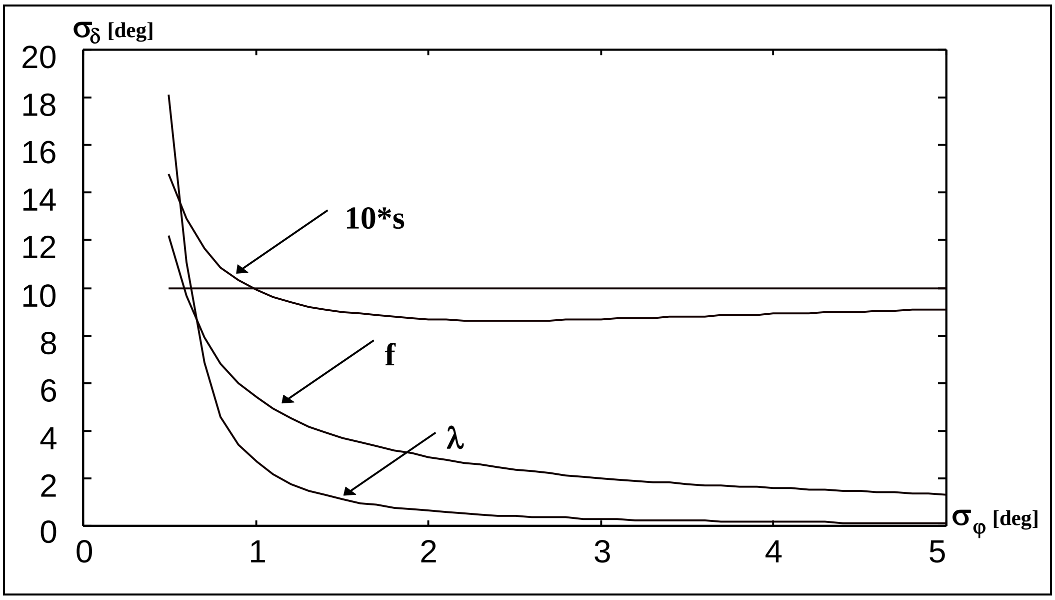

Fig. 2. *Shadow price function*

5 CONCLUSION

The proposed method provides the means for making the analysis of desired performances of a control system, set by the designer according to the total cost (energy consumption, control activity etc.). Sometimes it can be concluded that a slight relaxation of desired accuracy specifications (if technically sound) can result in significant total energy savings. The future research should investigate the interdependence between the parameters of the shadow price and the robustness of the control system. The preliminary analysis shows that some form of interdependence exists, because with a significant growth of the shadow price, the robustness of the control system deteriorates.

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